THE CONTACT PROBLEM OF AN ELASTIC LAYER COMPRESSED BY TWO PUNCHES

PMM Vol. 31, No. 4, 1967, pp. 711-715

Iu.N. KUZ'MIN and Ia.S. UFLIAND
(Leningrad)

(*Received March 6, 1967*)

The study is concerned with the deformation of an infinite elastic layer under the action of two rigid punches whose plane circular cross-sections differ from each other (Fig. 1). The problem is first reduced to a system of dual integral equations which are then transformed into a system of two regular Fredholm integral equations. The solution of the latter system is given in the form of a power series in a/h. Relations are also obtained between punch displacements and the applied loads.

1. **Formulation of the problem and its reduction to a system of integral equations.** We will utilize the known expressions for displacements and stresses in terms of two Papkovich-Neuber harmonic functions

$$2Gu_z = (3-4\nu)\Phi - F - z\frac{\partial\Phi}{\partial z} \tag{1.1}$$

$$\sigma_z = 2(1-\nu)\frac{\partial\Phi}{\partial z} - \frac{\partial F}{\partial z} - z\frac{\partial^2\Phi}{\partial z^2}, \quad \tau_{rz} = \frac{\partial}{\partial r}\left[(1-2\nu)\Phi - F - z\frac{\partial\Phi}{\partial z}\right] \tag{1.2}$$

Here G is the shear modulus and ν is Poisson's ratio (*).

If the axial displacements of the punches δ_a and δ_b are assumed to be known, then, in the absence of friction, the problem consists of the determination of two harmonic functions Φ and F in the regions $0 \leqslant r \leqslant \infty$, $-h < z < 0$, the functions satisfying the conditions

$$u_z = -\delta_a, \quad r < a; \quad \sigma_z = 0, \quad r > a; \quad \tau_{rz} = 0, \quad 0 \leqslant r < \infty; \quad z = 0 \tag{1.3}$$

$$u_z = \delta_b, \quad r < b; \quad \sigma_z = 0, \quad r > b; \quad \tau_{rz} = 0, \quad 0 \leqslant r < \infty; \quad z = -h \tag{1.4}$$

Moreover, for $r \to \infty$, the functions Φ and F must approach zero. We will seek a solution in the form of Hankel integral transform representations (**)

$$\Phi = \int_0^\infty [A \operatorname{sh}\lambda(z+h) + B \operatorname{ch}\lambda(z+h)] J_0(\lambda r)\frac{d\lambda}{\operatorname{sh}\lambda h}$$

$$F = \int_0^\infty \left\{[C \operatorname{sh}\lambda(z+h) + D \operatorname{ch}\lambda(z+h)]\frac{J_0(\lambda r)}{\operatorname{sh}\lambda h} + S(\lambda h)\right\} d\lambda \tag{1.5}$$

The conditions concerning the absence of shear stresses on the layer boundaries will be satisfied by the relations

$$(1-2\nu)(A + B\operatorname{cth}\mu) - C - D\operatorname{cth}\mu = 0$$

$$(1-2\nu)B - D + \mu A = 0, \quad \mu = \lambda h \tag{1.6}$$

---

*) For formulas involving the quantities u_r, σ_r and σ_φ, which do not appear in the boundary conditions, see, for example, [1], p. 230.

**) For the behavior of $S(\mu)$ see (1.18). It can be shown that the choice of $S(\mu)$ is essentially related to the condition $u_z \to 0$ for $r \to \infty$.

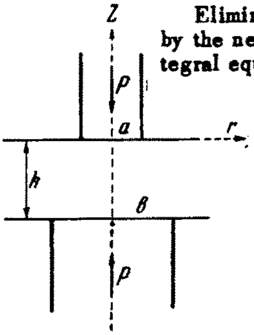

Fig. 1

Elimination of C and D with the aid of (1.6) and replacement of A and B by the new unknowns $M(\mu)$ and $N(\mu)$ leads to the following system of dual integral equations for the remaining boundary conditions

$$\int_0^\infty \left[P(\mu) J_0(\lambda r) - \frac{S(\mu)}{2(1-\nu)}\right] d\lambda = -\frac{G\delta_a}{1-\nu} \quad (0 \leqslant r < a) \quad (1.7)$$

$$\int_0^\infty \left[R(\mu) J_0(\lambda r) - \frac{S(\mu)}{2(1-\nu)}\right] d\lambda = \frac{G\delta_b}{1-\nu} \quad (0 \leqslant r < b) \quad (1.8)$$

$$\int_0^\infty M(\mu) J_0(\lambda r)\, d\lambda = 0, \quad a < r < \infty;$$

$$\int_0^\infty N(\mu) J_0(\lambda r)\, d\lambda = 0, \quad b < r < \infty \quad (1.9)$$

The following notation has been introduced in (1.7) and (1.8):

$$P = A + B \operatorname{cth} \mu = \frac{M-N}{\mu} T + M \frac{1-Q}{\mu}, \; Q = \frac{1+\mu-e^{-\mu}}{\operatorname{sh}\mu + \mu}$$

$$R = \frac{B}{\operatorname{sh}\mu} = \frac{M-N}{\mu} T - N \frac{1-Q}{\mu}, \quad T = \frac{\operatorname{sh}\mu + \mu \operatorname{ch}\mu}{\operatorname{sh}^2\mu - \mu^2}$$

The substitution (see, for example, [1], Chapter VIII)

$$M(\mu) = \mu \int_0^a \varphi(t) \cos \lambda t\, dt, \quad N(\mu) = \mu \int_0^b \psi(t) \cos \lambda t\, dt \quad (1.11)$$

makes it possible to satisfy (1.9) and to reduce (1.7) and (1.8) to the form

$$\int_0 \frac{\varphi(t)\, dt}{\sqrt{r^2-t^2}} = -\frac{G\delta a}{1-\nu} + \int_0^a \varphi(t)\, dt \int_0^\infty Q(\mu) J_0(\lambda r) \cos \lambda t\, d\lambda - \quad (1.12)$$

$$-\int_0^\infty \left\{\left[\int_0^a \varphi(t) \cos \lambda t\, dt - \int_0^b \psi(t) \cos \lambda t\, dt\right] T(\mu) J_0(\lambda r) - \frac{S(\mu)}{2(1-\nu)}\right\} d\lambda, \quad 0 \leqslant r < a$$

$$\int_0^r \frac{\psi(t)\, dt}{\sqrt{r^2-t^2}} = -\frac{G\delta_b}{1-\nu} + \int_0^b \psi(t)\, dt \int_0^\infty Q(\mu) J_0(\lambda r) \cos \lambda t\, d\lambda + \quad (1.13)$$

$$+\int_0^\infty \left\{\left[\int_0^a \varphi(t) \cos \lambda t\, dt - \int_0^b \psi(t) \cos \lambda t\, dt\right] T(\mu) J_0(\lambda r) - \frac{S(\mu)}{2(1-\nu)}\right\} d\lambda, \quad 0 \leqslant r < b$$

For further transformation, we formulate the static equilibrium conditions

$$-P = \int_0^a \int_0^{2\pi} \sigma_z |_{z=0} r\, dr\, d\varphi = \int_0^b \int_0^{2\pi} \sigma_z |_{z=-h} r\, dr\, d\varphi \quad (1.14)$$

Taking into account Expressions

$$\sigma_z |_{z=0} = \frac{1}{h} \int_0^\infty M(\mu) J_0(\lambda r)\, d\lambda, \quad \sigma_z |_{z=-h} = \frac{1}{h} \int_0^\infty N(\mu) J_0(\lambda r)\, d\lambda \quad (1.15)$$

as well as the relations [2]

$$\int_0^\infty J_1(\lambda a)\cos\lambda t\,d\lambda=\frac{1}{a},\qquad t<a \tag{1.16}$$

(1.14) may be reduced to the form

$$\int_0^a \varphi(t)\,dt=\int_0^b \psi(t)\,dt=-\frac{P}{2\pi} \tag{1.17}$$

If we now set

$$S(\mu)=2(1-\nu)\,T(\mu)\left[\int_0^a \varphi(t)\cos\lambda t\,dt-\int_0^b \psi(t)\cos\lambda t\,dt\right] \tag{1.18}$$

and utilize (1.17), then we may, by inverting the order of integration in (1.12) and (1.13), obtain the following system of equations for the unknown functions $\varphi(t)$ and $\psi(t)$:

$$\int_0^r \frac{\varphi(t)\,dt}{\sqrt{r^2-t^2}}=-\frac{G\delta_a}{1-\nu}+\int_0^a \varphi(t)\,dt\int_0^\infty Q(\mu)J_0(\lambda r)\cos\lambda t\,d\lambda+ \tag{1.19}$$

$$+\int_0^a \varphi(t)\,dt\int_0^\infty T(\mu)(1-\cos\lambda t)[1-J_0(\lambda r)]\,d\lambda-$$

$$-\int_0^b \psi(t)\,dt\int_0^\infty T(\mu)(1-\cos\lambda t)[1-J_0(\lambda r)]\,d\lambda,\qquad 0\leqslant r<a$$

$$\int_0^r \frac{\psi(t)\,dt}{\sqrt{r^2-t^2}}=-\frac{G\delta_b}{1-\nu}+\int_0^b \psi(t)\,dt\int_0^\infty Q(\mu)J_0(\lambda r)\cos\lambda t\,d\lambda+ \tag{1.20}$$

$$+\int_0^b \psi(t)\,dt\int_0^\infty T(\mu)(1-\cos\lambda t)[1-J_0(\lambda r)]\,d\lambda-$$

$$-\int_0^a \varphi(t)\,dt\int_0^\infty T(\mu)(1-\cos\lambda t)[1-J_0(\lambda r)]\,d\lambda,\qquad 0\leqslant r<b$$

Solving (1.19) and (1.20) as Schlömilch integral equations with known right-hand sides and making use of Eq.

$$\frac{d}{dt}\int_0^t [1-J_0(\lambda r)]\frac{r\,dr}{\sqrt{t^2-r^2}}=1-\cos\lambda t \tag{1.21}$$

we obtain a system of Fredholm integral Eqs.

$$\varphi(t)=-\frac{2G\delta_a}{\pi(1-\nu)}+\frac{2}{\pi}\int_0^a [L(t,\tau)+K(t,\tau)]\,\varphi(\tau)\,d\tau-$$

$$-\frac{2}{\pi}\int_0^b K(t,\tau)\,\psi(\tau)\,d\tau,\qquad 0\leqslant t<a \tag{1.22}$$

$$\psi(t)=-\frac{2G\delta_b}{\pi(1-\nu)}+\frac{2}{\pi}\int_0^b [L(t,\tau)+K(t,\tau)]\,\psi(\tau)\,d\tau-$$

$$-\frac{2}{\pi}\int_0^a K(t,\tau)\,\varphi(\tau)\,d\tau,\qquad 0\leqslant t<b \tag{1.23}$$

whose symmetric kernels are given by Formulas

$$L(t,\tau)=\int_0^\infty Q(\mu)\cos\lambda t\cos\lambda\tau\,d\lambda,\quad K(t,\tau)=\int_0^\infty T(\mu)(1-\cos\lambda t)(1-\cos\lambda\tau)\,d\lambda \tag{1.24}$$

2. Determination of the relations between punch displacements and applied loads. For convenience, Eqs. (1.22) and (1.23) may be divided into two systems of integral Eqs.

$$\begin{aligned}\omega_1(x)&=1+\varepsilon\int_0^1(R+S)\,\omega_1(y)\,dy\,dy-\varepsilon\int_0^\gamma S\omega_3(y)\,dy\\ \omega_3(x)&=\varepsilon\int_0^\gamma(R+S)\,\omega_3(y)\,dy-\varepsilon\int_0^1 S\omega_1(y)\,dy\end{aligned}\tag{2.1}$$

$$\begin{aligned}\omega_2(x)&=\varepsilon\int_0^1(R+S)\,\omega_2(y)\,dy-\varepsilon\int_0^\gamma S\omega_4(y)\,dy\\ \omega_4(x)&=1+\varepsilon\int_0^\gamma(R+S)\,\omega_4(y)\,\mathrm{dy}-\varepsilon\int_0^1 S\omega_2(y)\,dy\end{aligned}\tag{2.2}$$

wherein the following dimensionless quantities have been introduced:

$$\varepsilon=\frac{a}{h},\qquad\gamma=\frac{b}{a},\qquad x=\frac{t}{a},\qquad y=\frac{\tau}{a}\tag{2.3}$$

$$\omega_1(x)+\frac{\delta_b}{\delta_a}\omega_2(x)=-\frac{\pi(1-\nu)}{2G\delta_a}\varphi(t),\qquad \omega_3(x)+\frac{\delta_b}{\delta_a}\omega_4(x)=-\frac{\pi(1-\nu)}{2G\delta_a}\psi(t)\tag{2.4}$$

$$R=R(x,y,\varepsilon)=\frac{2}{\pi}\int_0^\infty Q(\mu)\cos\mu\varepsilon x\cos\mu\varepsilon y\,d\mu\tag{2.5}$$

$$S=S(x,y,\varepsilon)=\frac{2}{\pi}\int_0^\infty T(\mu)(1-\cos\mu\varepsilon x)(1-\cos\mu\varepsilon y)\,d\mu\tag{2.6}$$

Now, (1.17) may be transformed into the following Eqs: (2.7)

$$P=\frac{4Ga}{1-\nu}\left[\delta_a\int_0^1\omega_1(x)\,dx+\delta_b\int_0^1\omega_2(x)\,dx\right],\qquad P=\frac{4Ga}{1-\nu}\left[\delta_a\int_0^\gamma\omega_3(x)\,dx+\delta_b\int_0^\gamma\omega_4(x)\,dx\right]$$

from which the desired expressions for the punch displacements are obtained (2.8)

$$\delta_a=\frac{(1-\nu)P}{4Ga\Delta}\left[\int_0^\gamma\omega_4(x)\,dx-\int_0^1\omega_2(x)\,dx\right],\qquad \delta_b=\frac{(1-\nu)P}{4Ga\Delta}\left[\int_0^1\omega_1(x)\,dx-\int_0^\gamma\omega_3(x)\,dx\right]$$

$$\Delta=\int_0^1\omega_1(x)\,dx\int_0^\gamma\omega_4(x)\,dx-\int_0^1\omega_2(x)\,dx\int_0^\gamma\omega_3(x)\,dx\tag{2.9}$$

In order to obtain actual results for various values of the geometric parameters $\varepsilon = a/h$ and $\gamma = b/a$, it is necessary to employ some numerical method for the determination of the functions $\omega_k(x)$ $(k = 1, 2, 3, 4)$ in (2.1) and (2.2).

For small values of ε, we may obtain simple relations by expanding the kernels R and S and the unknown functions ω_k in power series of ε

$$R=\sum_{n=0}^\infty R_{2n}\varepsilon^{2n},\qquad S=\sum_{n=0}^\infty S_{2n}\varepsilon^{2n},\qquad \omega_k=\sum_{n=0}^\infty \omega_k^{(n)}\varepsilon^n\tag{2.10}$$

Expressions for the first-term coefficients in the indicated series are given by

$$R_0 = \frac{2r_0}{\pi}, \quad R_2 = -\frac{r_2}{\pi}(x^2+y^2), \quad R_4 = \frac{r_4}{12\pi}(x^4+6x^2y^2+y^4)$$

$$S_0 - S_2 = 0, \quad S_4 = \frac{\sigma_4}{2\pi}x^2y^2, \quad \omega_1^{(0)} = 1, \quad \omega_1^{(1)} = \frac{2r_0}{\pi}, \quad \omega_1^{(2)} = \left(\frac{2r_0}{\pi}\right)^2$$

$$\omega_1^{(3)} = \left(\frac{2r_0}{\pi}\right)^3 - \frac{r_2}{\pi}\left(x^2+\frac{1}{3}\right), \qquad \omega_1^{(4)} = \left(\frac{2r_0}{\pi^2}\right)^4 - \frac{2}{\pi}(x^2+1)r_0r_2$$

$$\omega_1^{(5)} = \left(\frac{2r_0}{\pi}\right)^5 - \frac{4}{\pi^3}\left(x^2+\frac{5}{2}\right)r_0^2r_2 + \frac{r_4}{12\pi}\left(x^4+2x^2+\frac{1}{5}\right) + \frac{\sigma_4}{6\pi}x^2$$

$$\omega_k^{(n)} \equiv 0 \quad (k=2,3;\ n=0,1,2,3,4), \qquad \omega_2^{(5)} = -\frac{\sigma_4}{6\pi}\gamma^3x^2, \qquad \omega_3^{(5)} = -\frac{\sigma_4}{6\pi}x^2$$

$$\omega_4^{(0)} = 1, \qquad \omega_4^{(1)} = \frac{2r_0\gamma}{\pi}, \qquad \omega_4^{(2)} = \left(\frac{2r_0\gamma}{\pi}\right)^2$$

$$\omega_4^{(3)} = \left(\frac{2r_0\gamma}{\pi}\right)^3 - \frac{r_2\gamma}{\pi}\left(x^2+\frac{\gamma^2}{3}\right), \qquad \omega_4^{(4)} = \left(\frac{2r_0\gamma}{\pi}\right)^4 - \frac{2\gamma^2}{\pi^2}(x^2+\gamma^2)r_0r_2$$

$$\omega_4^{(5)} = \left(\frac{2r_0\gamma}{\pi}\right)^5 - \frac{4\gamma^3}{\pi^3}\left(x^2+\frac{5}{3}\gamma^2\right)r_0^2r_2 + \frac{r_4\gamma}{12\pi}\left(x^4+2x^2\gamma^2+\frac{\gamma^4}{5}\right) + \frac{\sigma_4}{6\pi}x^2\gamma^3$$

$$r_m = \int_0^\infty Q(\mu)\mu^m\,d\mu, \qquad \sigma_m = \int_0^\infty T(\mu)\mu^m\,d\mu \tag{2.11}$$

The approximations for (2.8) and (2.9) are given by

$$\delta_a = \frac{(1-\nu)P}{4Ga}\left[1 - \frac{2r_0}{\pi}\varepsilon + \frac{2r_2}{3\pi}\varepsilon - \left(\frac{\sigma_4}{18\pi}+\frac{4r_4}{45\pi}\right)\varepsilon^5 + \frac{\sigma_4}{18\pi}\gamma^2\varepsilon^5 + O(\varepsilon^7)\right] \tag{2.12}$$

$$\delta_b = \frac{(1-\nu)P}{4Gb}\left[1 - \frac{2r_0}{\pi}\gamma\varepsilon + \frac{2r_2}{3\pi}\gamma^3\varepsilon^3 - \left(\frac{\sigma_4}{18\pi}+\frac{4r_4}{45\pi}\right)\varepsilon^5\gamma^5 + \frac{\sigma_4}{18\pi}\gamma^3\varepsilon^5 + O(\varepsilon^7)\right] \tag{2.13}$$

Formula (2.13) may be obtained from (2.12) by interchanging a and b, as one might expect.

Application of a limiting process to (2.12) for $h \to \infty$ ($\varepsilon \to 0$) yields the known relation

$$\delta = \frac{(1-\nu)P}{4Ga}$$

which holds for the case of a single punch acting on a half-space. The next three terms in (2.12) which are independent of $\gamma = b/a$ characterize the effect of the $z = -h$ plane, which is unstressed, but is subjected to the axial load P. Finally, the last term in (2.12) take into account the application of the second punch with finite radius b to the $z = -h$ plane.

Evaluating the integrals in (2.11) we obtain

$$r_0 = 2.335, \qquad r_2 = 12.65, \qquad r_4 = 262.2, \qquad \sigma_4 = 296.8$$

whereupon (2.12) yields

$$\delta_a = \frac{(1-\nu)P}{4Ga}\left[1 - 1.49\varepsilon + 2.68\varepsilon^3 - 12.67\varepsilon^5 + 5.25\gamma^2\varepsilon^5 + O(\varepsilon^7)\right] \tag{2.14}$$

In particular, for $\varepsilon = 1/4$, we have

$$\frac{4Ga\delta_a}{(1-\nu)P} \approx 0.657 + 0{,}00510\ \gamma^2 \tag{2.15}$$

The last expression shows that under these conditions the influence of the size of the second punch becomes appreciable only for substantial values of $\gamma = b/a$ (of order 3 – 4).

In conclusion, we note that for $\gamma = 1$ the problem under consideration may, from symmetry considerations, be treated like a punch of radius a acting on an elastic layer of thickness $\frac{1}{2}h$ resting without friction on a rigid substrate. The solution to this problem was obtained in [3] by means of dual integral equations, and, for the case of $\gamma = 1$, Formula (2.15) yields $4Ga\delta_a = 0.662\,(1-\nu)P$, which coincides with the corresponding result in [3] (cf. Table 3 for $p = 0.5$).

BIBLIOGRAPHY

1. Ufliand, Ia.S., Integral Transforms in Elasticity Problems. M.-L., Izd. Akad. Nauk SSSR, 1963.
2. Watson, G.N., Theory of Bessel Functions. M., Izd. inostr. lit., 1949.
3. Lebedev, N.N. and Ufliand, Ia.S., Axisymmetric contact problem for an elastic layer. PMM, Vol. 22, No. 3, 1958.

Translated by H.H.

ON THE POSSIBILITY OF SOLVING PLATE STABILITY PROBLEMS WITHOUT A PRELIMINARY DETERMINATION OF THE INITIAL STATE OF STRESS

PMM Vol. 31, No. 4, 1967, pp. 716-722

N.A. ALFUTOV and L.I. BALABUKH
(Moscow)

(*Received January 2, 1967*)

Initial stresses in the middle plane enter into the differential equation for plate stability and the corresponding Bryan energy criterion.

In the general case these stresses are determined from the solution of the plane problem. For a plate subjected to complex contour loadings, concentrated forces, for example, the solution of this problem is very complicated.

At the same time, the buckling energy criterion allows a representation in which only the work of the given external forces enters, in addition to the potential energy of plate bending. Hence, the natural question arises as to whether it is generally necessary to know the distribution of the true initial stresses in solving stability problems. It is shown herein that critical values of the external loadings may be found without determining the initial state of stress of the plate.

A new form of the buckling energy criterion is obtained in which the initial stresses do not enter. It is shown that in determining the additional tangential displacements in which the external loadings do work in plate buckling, it is impossible, in the general case, to utilize conditions of inextensibility of the middle plane.

The proposed method of determining the critical loadings without a preliminary solution of the plane problem is illustrated by examples. The known Sommerfeld problem of stability of a rectangular plate compressed by concentrated forces is considered.

1. Let u, v, w be the components of the total displacement vector of points of the middle plane of the plate in a rectangular x, y, z coordinate system. The x and y axes lie in the plane of the plate. The strains in the middle plane of the plate are

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \quad (1.1)$$

We consider the stresses in the middle plane to satisfy the equilibrium Eqs.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = 0 \quad (1.2)$$

and therefore

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2}, \qquad \tau = -\frac{\partial^2 \varphi}{\partial x\, \partial y} \quad (1.3)$$

The stresses on the plate contour are connected with the loadings X and Y by means of the dependences